Application of the S_n Method to Spherically Symmetric Radiative-Transfer Problems

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The model of two concentric spheres separated by an absorbing and emitting medium is employed to study the applicability of the S_n method, commonly used in neutron-transport theory, to radiative-transfer calculations. Equilibrium and nonequilibrium radiative fields, with a wide range of variations of the radius ratio of the two spheres, are considered for the general assessment of the accuracy of the S_n method. Two different schemes have been used to cope with the problem of numerical instability involved in the integration of governing equations. Results for the simulation of re-entry problems and other applications indicate that the S_n method is a convergent and accurate technique suitable for radiative-transfer calculations. For the cases considered, the S_4 method is practically exact and the S_3 method will be generally sufficient for most engineering applications.

Introduction

THE multidimensional aspect of radiative transport needs to be considered to investigate realistically the coupling between radiative, convective, and ablative processes in flows around entry bodies. The principal difficulty of such an analysis is in properly handling the angular variation of radiation. An exact formulation of the problem results in an integro-differential equation, which has been solved only in special cases. The motivation for this study is to establish the validity of an approximate method that is extendable to fully coupled re-entry problems, including a multidimensional nongray radiative-transfer analysis.

A versatile technique commonly used in neutron-transport theory is the so-called S_n method.[1-6] The technique is quite flexible and is capable of handling multidimensional problems.[3-6] The purpose of this paper is to study the applicability of the S_n method to radiative-transfer calculations. To this end, we consider a simple physical model of two concentric spheres, of radii r_1 and r_2, separated by an absorbing and emitting medium with internal heat generation. The model is chosen because 1) it simulates the blunt-body problem near its stagnation point, 2) the differential approximation[7] is known to be inadequate,[8-11] and 3) the exact solution[12-16] is available for comparison. This relatively simple problem has also been chosen by several other investigators as the test problem of their approximate methods.[9-11,17-19] However, it is difficult or unclear as to how their techniques can be applied to more general situations.

The accuracy of the S_n method for this simple test problem will be assessed by comparing solutions obtained by this method with exact and approximate results for both equilibrium and nonequilibrium radiative fields. While only those cases of r_1/r_2 close to unity will be of direct interest to re-entry problems, a wide range of variation of this parameter will be considered not only because this will give a more complete assessment of the S_n method in nonplanar situations, but also because the problem of radiative transfer in spherical geometries is of interest in astrophysical applications as well as in studies of spherical blast waves due to strong explosions or laser-induced micro-explosions for which r_1/r_2 will not necessarily be close to 1.[20-21]

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Assumptions and Basic Formulation

The geometrical configuration of the problem is shown in Fig. 1. Two concentric spheres, of radii r_1 and r_2, are separated by a stagnant medium which is nonconducting, nonscattering, and in local thermodynamic equilibrium. In order to compare with available results, we make the additional simplification that the volumetric absorption coefficient α is gray and independent of temperature. We emphasize that the method is easily extendable to the more general case of nongray, variable absorption coefficients. With the above assumptions, the radiative-transfer equation for the specific intensity $I(\mu, \tau)$ is[22]

$$\mu\partial I/\partial\tau + [(1-\mu^2)/\tau]\partial I/\partial\mu = B - I \qquad (1)$$

where μ is the direction cosine, $\mu = \cos\theta$; τ the optical distance, $\tau = \alpha r$; and B the Planck's function, $B = \sigma T^4/\pi$. Mathematically speaking, Eq. (1) is a partial differential equation for I in terms of the independent variables μ and τ. The basic idea of the S_n method is to divide the interval $-1 \le \mu \le 1$ into n (not necessarily even) subintervals, $\mu_{j-1} \le \mu \le \mu_j$, $j = 1, 2, \ldots, n$, (where $\mu_o = -1$ and $\mu_n = 1$), and approximate the specific intensity $I(\mu, \tau)$ by n connected straight-line segments as follows

$$I(\mu, \tau) = \frac{1}{(\mu_j - \mu_{j-1})}[(\mu - \mu_{j-1})I_j(\tau) + (\mu_j - \mu)I_{j-1}(\tau)] \qquad (2)$$

where $I_j(\tau) \equiv I(\mu_j, \tau)$. We substitute Eq. (2) into Eq. (1) and then integrate both sides of Eq. (1) over μ from μ_{j-1} to μ_j. For μ_j independent of τ there results

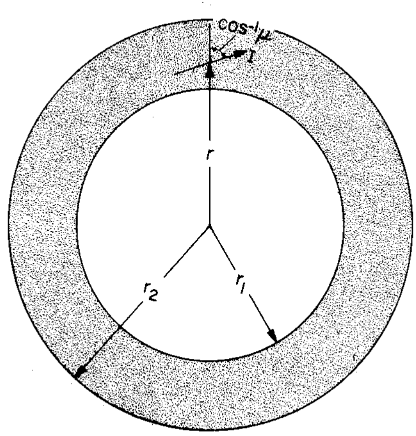

Fig. 1 Geometrical configuration.

$$\left(a_j\frac{d}{d\tau}+\frac{b_j}{\tau}+1\right)I_j+\left(\bar{a}_j\frac{d}{d\tau}-\frac{b_j}{\tau}+1\right)I_{j-1}=2B, \quad (3)$$

$$j=1,2,\ldots,n$$

where

$$a_j=(2\mu_j+\mu_{j-1})/3$$
$$\bar{a}_j=(\mu_j+2\mu_{j-1})/3$$

$$b_j=\frac{2(3-\mu_j^2-\mu_j\mu_{j-1}-u_{j-1}^2)}{3(\mu_j-\mu_{j-1})}$$

Since there are $n+1$ variables, an additional equation is obviously required. This relation is obtained by setting $\mu=-1$ in Eq. (1)

$$dI_o/d\tau=I_o-B \quad (4)$$

The mean intensity $I^0(\equiv 2\pi\int_{-1}^{1} Id\mu)$ and the radiative flux $q(\equiv 2\pi\int_{-1}^{1}\mu Id\mu)$ are related to the I_j's by

$$I^0=\pi\sum_{j=1}^{n}(\mu_j-\mu_{j-1})(I_j+I_{j-1}) \quad (5)$$

$$q=\pi\sum_{j=1}^{n}(\mu_j-\mu_{j-1})(a_jI_j+\bar{a}_jI_{j-1}) \quad (6)$$

It is worth noting that Eqs. (3) and (4) can be considered as a finite-difference approximation to the equation of radiative-transfer with a rather flexible choice of μ_j. In addition, Eq. (3) involves only two adjacent dependent variables I_j and I_{j-1}. With I_0 solved from Eq. (4), I_j can be determined successively in increasing j from Eq. (3).

The interval $-1\leqslant\mu\leqslant 1$ is divided into n uneven intervals in the following way. As Chou and Tien[10] have noted, for any point, τ, in the annular gap the limiting position of the ray starting from the inner sphere that can reach this point is given by (Fig. 1)

$$\mu=\mu^*\equiv[1-(\tau_1/\tau)^2]^{1/2} \quad (7)$$

that was used as one of the natural dividing points in their analysis. Since $d\mu^*/d\tau$ is unbounded at $\tau=\tau_1$, μ^* cannot be used directly in the S_n method. Instead, we use the mean value of μ^* at the two end points $\tau=\tau_1$ and τ_2,

$$\bar{\mu}=\{[1-(\tau_1/\tau_2)^2]^{1/2}\}/2 \quad (8)$$

as one of the dividing points in the formulation. We considered an S_3 and an S_4 method with the μ-interval divided as $(-1, 0, \bar{\mu}, 1)$ and $(-1, -\bar{\mu}, 0, \bar{\mu}, 1)$, respectively. We note that the incorporation of this particular geometrical aspect of the problem in the formulation is not essential to the S_n method. In fact, an S_4 method with even μ-division has also been considered and the result so obtained differs by about three percent at most from the corresponding S_4 method with uneven μ-division. All of the results to be presented under Results and Discussion are obtained from the uneven μ-division.

Boundary Conditions

For simplicity, we consider only the case with black walls. The exact boundary conditions are at $\tau=\tau_1$

$$I=B(T_{w_1})\equiv B_{w_1} \text{ for } \mu\geq 0 \quad (9a)$$

and at $\tau=\tau_2$

$$I=B(T_{w_2})\equiv B_{w_2} \text{ for } \mu\leq 0 \quad (9b)$$

The corresponding discrete representation at $\tau=\tau_2$ is

$$I_j=B_{w_2}, 0\leq j\leq m \quad (10a)$$

where $\mu_m=0$. Since there are $n+1$ first-order ordinary differential equations, only n-m boundary conditions can be imposed at $\tau=\tau_1$. This does not present a problem for a one-sphere case, $\tau_1=0$, as was the case considered by Carlson,[1] nor for a perfectly reflecting wall at $\tau=\tau_1$. However, for a black wall, since the corresponding discrete representation of Eq. (9a) involves n-$m+1$ variables, the exact boundary condition can be satisfied only approximately. Good results were obtained by satisfying the first n-m half-range moments of Eq. (9a)

$$\int_0^1\mu^i I(\mu,\tau_1)\,d\mu=B_{w_1}/(i+1), \quad i=0,1,\ldots,n-m-1 \quad (10b)$$

Energy Conservation

In order to have a rather general assessment of the accuracy of the S_n method, we consider the following two separate classes of problems.

A. Internal Source Distribution Prescribed

In this first class, we prescribe the distribution of the internal heat source. As Olfe[9] and Viskanta and Crosbie[15] have noted, the solution for the general case of radiative transfer between two concentric spheres of different temperatures with uniform internal heat generation may be obtained as a superposition of two problems: 1) black walls at different temperatures, no internal heat generation, and 2) black walls at equal temperatures, uniform internal heat generation per unit volume S. The equation of energy conservation for both problems can be written as

$$4\pi B-I^\circ=S/\alpha\equiv\bar{s}=\text{const} \quad (11)$$

For the first problem, $\bar{s}=0$, and the system is in radiative equilibrium, that is, the steady-state situation of the system is maintained by a balance between the emission and the absorption processes.

B. Gas Temperature Distribution Prescribed

The gas temperature distribution obtained from the first class of problems seldom exists in re-entry problems because of the presence of coupling between radiative and other (e.g., convective) energy-transport mechanisms. In order to have a better assessment of the S_n method, we simulate such situations by prescribing the temperature distribution of the gas between the concentric spheres and use the computed values of heat flux and flux divergence as the basis for comparison. An analytic expression for the exact value of the inner wall flux, q_w, was given by Kennet and Strack[12] for a medium of uniform temperature between cold walls. Chisnell[16] extended the work and presented results of q_w obtained by the exact, the tangent slab and the differential approximation methods for cold walls with the Planck's function distribution of the gas as

$$B(\tau)/B(\tau_2)=[(\tau-\tau_1)/(\tau_2-\tau_1)]^P,\ P=0,1,2 \quad (12)$$

for several values of τ_1/τ_2. A more reasonable simulation of a radiation-cooled shock layer would be a linear temperature distribution with a non-negligible temperature at the inner wall:

$$T(\tau)/T(\tau_2)=1-\beta[(\tau_2-\tau)/(\tau_2-\tau_1)] \quad (13)$$

From a coupled, nongray, shock-layer calculation, β is typically[23] about 0.3. We realize, of course, that a linear temperature distribution will be unrealistic when the gas layer becomes optically thick.

Methods of Solution

Two different numerical schemes have been used to solve the governing Eqs. 3, 4, and 11 or 13 with boundary conditions (10a) and (10b). As Carlson[1] has noted, numerical instability will occur if the integration in τ is not in the direction of photon (or neutron) flow. In other words, to obtain accurate solutions, we should integrate the equations with $a_j > 0$ in the increasing τ direction and conversely with $a_j < 0$.[4] However, if numerical instability is not strong enough to destroy the accuracy of the result when all equations are integrated in one direction, it may be advantageous to do so for the case when the internal source distribution is prescribed. This is because $B(\tau)$ is related to all of the I_j's by Eqs. (5) and (11). Since the governing equations are linear, superposition of complementary and particular solutions obtained by numerically integrating the equations with the standard Runge-Kutta scheme can be used to generate the desired solution. The method is very efficient since in principle no iteration will be required. In practice, however, we perturb about the obtained solution to improve upon the accuracy and to reduce the numerical error propagation mentioned before. Typically, it takes about one minute for a solution on IBM 7094.

For τ_1/τ_2 small or $\tau_2 - \tau_1$ large or both compared to 1, the numerical difficulty discussed before becomes so strong as to render a solution by integrating all the equations in the same direction impossible. For these cases we choose to convert the governing system of equations into difference equations. Following Carlson[1] and Underhill and Russell,[4] we divide the range $\tau_1 \leq \tau \leq \tau_2$ into k intervals and assume that both I_j and B are linear in τ over each of the interval (τ_{j-1}, τ_i). Integrating Eq. (3) over τ from τ_{i-1} to τ_i, we obtain

$$\begin{aligned}(a_j+b_jg_i+h_i)I_{j,i}-(a_j+b_j\bar{g}_i-h_i)I_{j,i-1}\\+(\bar{a}_j-b_jg_i+h_i)I_{j-1,i}-(\bar{a}_j-b_j\bar{g}_i-h_i)I_{j-1,i-1}\\=2h_i(B_i+B_{i-1})\end{aligned} \tag{14}$$

where

$$\begin{aligned}I_{j,i} &\equiv I(\mu_j, \tau_i)\\B_i &\equiv B(\tau_i)\\h_i &= (\tau_i - \tau_{i-1})/2\\g_i &= 1 - (\tau_{i-1}/2h_i)\ \ln(\tau_i/\tau_{i-1})\\g_i &= (\tau_i/2h_i)\ \ln(\tau_i/\tau_{i-1})\end{aligned}$$

We note that if one is interested in a one-sphere problem, that is, $\tau_1 = 0$, the forms for g_i and $\bar{g}_i$ have to be modified.[4] However, we shall only consider the case $\tau_1 > 0$ here.

Eq. (4) can be converted in a similar manner to the following finite-difference form:

$$(1 - h_i)I_{o,i} - (1+h_i)I_{o,i-1}+h_i(B_i+B_{i-1}) = 0 \tag{15}$$

The algorithm for the problem goes as follows: Using Eq. (10a) as the starting value, Eq. (15) is used to compute $I_{o,i}$ from $\tau = \tau_2$ to $\tau = \tau_1$. We then can obtain $I_{j,i}$ successively in a similar manner from Eq. (14) for $j = 1, 2, \ldots, m$. With $I_{m,i}$ known, Eqs. (10b) and (14) can now be used to calculate $I_{j,i}$ from $\tau = \tau_1$ to $\tau = \tau_2$ for $j = m+1, \ldots, n$. For the case of prescribed internal source distribution, since $B(\tau)$ is related to all of the I_j's by Eqs. (5) and (11), some iterative procedure is necessary. This, of course, is not required for the case when the gas temperature distribution is prescribed. In general, when other energy transfer mechanisms are present, $B(\tau)$ will be coupled to other conservation equations and an iterative procedure will also be required. The basic idea involved in the iteration is very simple. First, an estimated distribution, $B_{\text{in}}(\tau)$, is used. Next, the radiative field is calculated and a new distribution, $B_{\text{out}}(\tau)$, is obtained from the energy Eq. (11). If the maximum difference between $B_{\text{in}}(\tau)$ and $B_{\text{out}}(\tau)$ is larger than a prescribed limit, a decision will be made as to how to obtain a revised $B_{\text{in}}(\tau)$. One simple algorithm is[24]

$$B_{\text{in}}^{(l+1)}(\tau) = (1 - f)\, B_{\text{in}}^{(l)}(\tau)+f\, B_{\text{out}}^{(l)}(\tau) \tag{16}$$

where the superscripts denote the number of iterations. In radiative transfer calculations coupled with flow considerations, a typical value of 0.5 is usually used for f.[23,25] For the present problem, $f = 1$ was found to be stable and to converge faster. In fact, an even better algorithm is to use Eq. (16) with $f = 1$ to compute the revised input data for a couple of times, then an extrapolation based on these input values is used to obtain a new input to start the next couple of iterations based on Eq. (16) with $f = 1$ again. This process is repeated until a converged solution is obtained. Typically, the solution time for the prescribed internal source distribution case is about five to ten minutes on the IBM 7094.

Results and Discussion

The errors of the approximate methods (in comparison with the exact solution of Viskanta and Crosbie[15]) in the radiative flux as a function of the optical thickness of the gas layer between the two concentric spheres are shown in Figs. 2 and 3. The cases are for radiative equilibrium (no internal heat generation). For this particular situation the total flux across any spherical surface, $4\pi r^2 q(\tau)$, is a constant. Therefore, the errors, as plotted, are the same for all values of τ. Shown in both figures are the results of S_3, S_4, the differential approximation and the 3-region solution of Chou and Tien[10] as improved by Hunt.[17] For the case of a small inner sphere, $\tau_1/\tau_2 = 0.1$ (Fig. 2), it is well known that the differential approximation predicts a flux value which is twice the exact quantity for the optically thin situation.[8,9] The results of the S_3, the S_4, and the 3-region methods all show an error of about one percent at most.

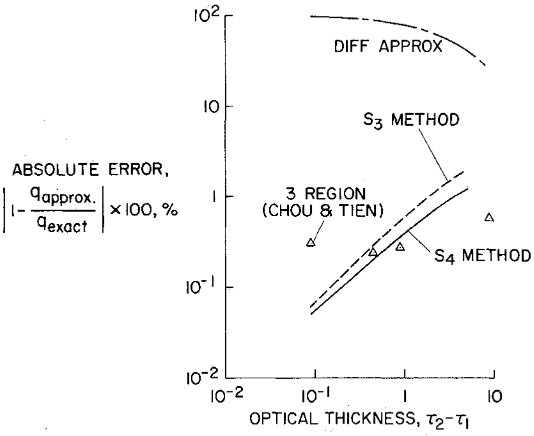

Fig. 2 Error analysis, radiative equilibrium, no internal heat generation, $r_1/r_2 = 0.1$.

A better representation in re-entry applications would be for a radius ratio of near 1. For $\tau_1/\tau_2 = 0.9$ (Fig. 3) the error of the differential approximation decreases to at most about ten percent, which is usually acceptable for most engineering applications. The other three methods give much smaller errors, especially

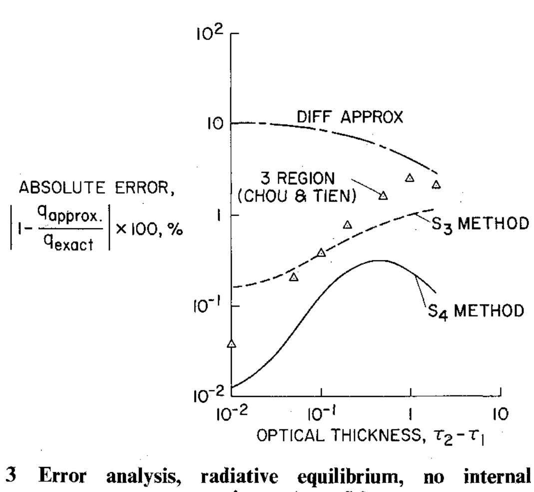

Fig. 3 Error analysis, radiative equilibrium, no internal heat generation, $r_1/r_2 = 0.9$.

for the optically thin case. Among all of the approximate methods shown, the S_4 method gives the best result.

Figure 4 shows the effect of the radius ratio on the fraction of heat radiated through the inner wall, Q_1/Q_{total}, where Q_{total} is the total flux going into both walls. The case is for uniform heat generation and $\tau_2 = 2$. For a slab geometry, $r_1/r_2 \rightarrow 1$, Q_1/Q_{total} would be 0.5 because of symmetry. The sharp decrease, as shown by the exact solution,[13] in Q_1/Q_{total} at $r_1/r_2 = 0.95$ from the planar value reflects the fact that since the gas layer is optically thin ($\tau_2 - \tau_1 = 0.1$ for this particular case) the geometrical effect is very apparent. The differential approximation predicts values that are too high. The 3-region solution gives erroneous results for $r_1/r_2 > 0.75$; the results from the S_3 method are somewhat high. This figure indicates again that the S_n method is convergent and the agreement between the exact and the S_4 method is very good.

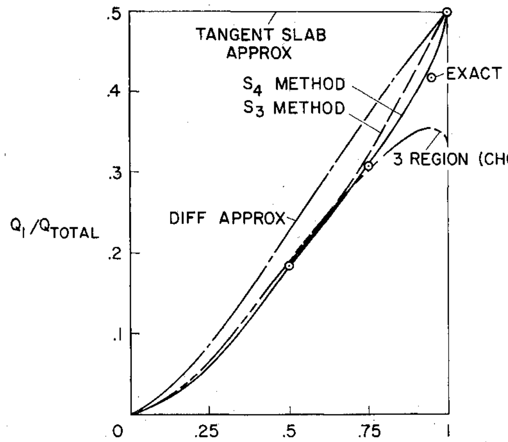

Fig. 4 Heat radiated through the inner wall as a function of the radius ratio, uniform internal heat generation, $\tau_2 = 2$.

The nondimensional emissive power distribution for different values of r_1/r_2 with uniform or no internal heat generation are shown in Ref. (26). It is again clear that the S_n method is convergent and the S_4 method gives essentially the exact result.

Figure 5 shows the error in the inner-wall flux as obtained by three different methods in the radiative nonequilibrium case of a linear gas-temperature distribution between cold walls. For a ratio of the gas-layer thickness to the (inner) radius of $\frac{1}{12}$, the tangent slab approximation, which neglects the curvature effect, introduces considerable error when the gas layer is optically thin. The differential approximation is the least accurate of all three approximate methods shown. The agreement between the S_4 method and the exact solution is excellent.

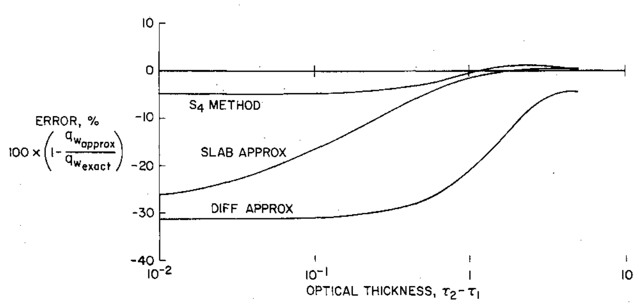

Fig. 5 Error in flux to the inner sphere for linear gas-temperature distribution between cold walls,
$(\tau_2-\tau_1)/\tau_1 = 1/12$, $T/T_2 = \mathbf{1.0-0.3}\,(\tau_2-\tau)/(\tau_2-\tau_1)$.

In an uncoupled calculation such as when the gas-temperature distribution is prescribed, a better comparison would be the flux divergence. This is because the divergence of the radiative heat flux vector in a coupled calculation represents a potential to change the gas temperature distribution. The exact normalized flux divergence and deviations from it as obtained by three different approximate methods are depicted in Figs. 6–8 for a linear gas-temperature distribution between cold walls. Since the exact flux divergence in the interior of the gass will approach zero when the gas layer becomes optically thick, differences instead of errors in percentage are shown in these figures.

For an optically thin gas (Fig. 6), the problem is essentially emission-dominated and the flux divergence value is close to the local pure emission limit, $4\alpha\sigma T^4$ All three approximate methods give quite good results. Among the approximate methods shown, the S_4 method is the most accurate.

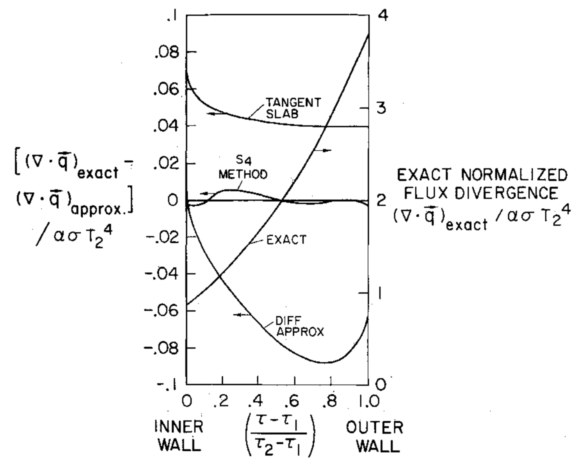

Fig. 6 Exact normalized flux divergence and deviations with linear gas-temperature distribution,
$(\tau_2-\tau_1)/\tau_1 = 1/12$, $\tau_2-\tau_1 = \mathbf{0.05}$.

As the optical thickness of the gas increases, the amount of radiative absorption builds up and the flux divergence decreases. For an optical thickness of order 1 (Fig. 7), we see that the differential approximation gives a maximum error of about 20%. Both the S_4 method and the tangent slab approximation give considerably better results.

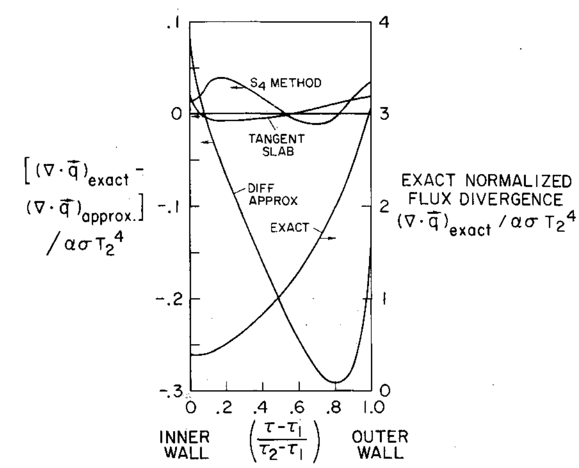

Fig. 7 Exact normalized flux divergence and deviations with linear gas-temperature distribution.
$(\tau_2-\tau_1)/\tau_1 = 1/12$, $\tau_2-\tau_1 = \mathbf{0.5}$.

As the gas layer becomes optically thicker, one would expect a smaller curvature effect and the tangent slab approximation to be more accurate. For an optical thickness of 5 (Fig. 8), the gas in the interior becomes close to radiative equilibrium. As was noted before, a linear gas temperature distribution becomes unrealistic for such an optically thick situation. Near the interior where the exact flux divergence is near zero, the S_4 method gives the smallest error among all three approximate methods shown.

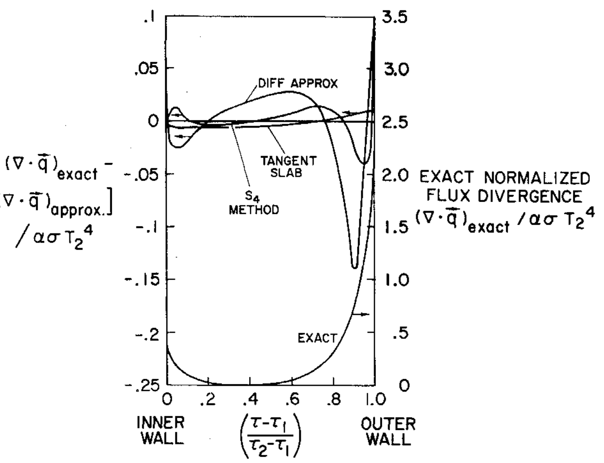

Fig. 8 Exact normalized flux divergence and deviations with linear gas-temperature distribution,
$(\tau_2-\tau_1)/\tau_1 = 1/12$, $\tau_2-\tau_1 = 5$.

Away from this region of near radiative equilibrium, however, the tangent slab approximation gives a somewhat better result. An S_5 calculation (not shown here) has also been performed and the result indicates a reduction in the overall error from the S_4 method by a factor of about 2.

Results for the corresponding case of $(\tau_2-\tau_1)/\tau_1 = 10$ (not shown here) have also been obtained. For this class of a small inner sphere problem, the S_4 method is far more accurate than either the tangent slab or the differential approximation. For practical purposes, the results of the S_4 method can be considered as exact.

Conclusions

The main conclusion we may draw from this study is that the S_n method is a convergent and accurate technique suitable for radiative-transfer calculations. For the cases considered, the S_4 method is practically exact and the S_3 method will be generally sufficient for most engineering applications. In addition, the method is very attractive because of the simplicity in concept, the ease in going to higher order approximations and the generally short computation time involved.

For the particular gas temperature distribution considered here to simulate re-entry problems near the stagnation point, the tangent slab approximation is reasonably good except for the optically thin case. Its adequacy in a more general situation, however, is uncertain. In view of the fact that exact solutions are not generally available, the convergent characteristic of the S_n method makes it a good candidate for radiative-transfer calculations in general multidimensional situations.

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